Pre-Calculus Notes

Sections 4.2 and 4.4 Meshed Together...

PART ONE:  We looked at trigonometric functions for angles measuring between 0° and 90° when we focused on the acute angles inside of a right triangle. We can also assign trig. values to ANY angle measure, including angles greater than 90°. But need to understand reference angles and reference triangles first.

First, let's review an important geometry concept...

Will the angle \( \theta \) have the same trig values REGARDLESS of the triangle used? _______ Why?

Second, let's look at reference angles and reference triangles.

A reference angle is the ACUTE angle (always positive) formed by the TERMINAL side of any angle in standard position and the nearest portion of the \( x \)-axis. The triangle formed is the reference triangle.

Third, let's start the associating the trig values with \( x \), \( y \), and \( r \), as well as \( \text{adj.} \), \( \text{opp.} \), and \( \text{hyp.} \).

Let \( \theta \) be an angle in standard position with \((x, y)\) a point on the terminal side of \( \theta \) and \( r = \sqrt{x^2 + y^2} \neq 0 \), since \( x^2 + y^2 = r^2 \). Then...

\[
\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{1}{\sin \theta} \\
\cos \theta = \frac{x}{r} \quad \text{and} \quad \sec \theta = \frac{1}{\cos \theta} \\
\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{1}{\tan \theta}
\]

Example 1:  Find the 6 trigonometric function values of the angle with a point \((3, 4)\) on the terminal side of the angle, \( \theta \).

\[
x = _____ \quad y = _____ \quad r = _____
\]

\[
\sin \theta = _____ \quad \cos \theta = _____ \quad \tan \theta = _____
\]

\[
\csc \theta = _____ \quad \sec \theta = _____ \quad \cot \theta = _____
\]
Example 2: Find the 6 trigonometric function values of the angle with a point \((-5, 12)\) on the terminal side of the angle, \(\theta\).

\[
\begin{align*}
x &= \_\_\_ \\
y &= \_\_\_ \\
r &= \_\_\_ \\
\sin \theta &= \_\_\_ \\
\cos \theta &= \_\_\_ \\
\tan \theta &= \_\_\_ \\
\csc \theta &= \_\_\_ \\
\sec \theta &= \_\_\_ \\
\cot \theta &= \_\_\_
\end{align*}
\]

Now let's look at determining the sign (positive or negative) of a function by looking at the quadrant in which the angle terminates.

The signs of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because \(\cos \theta = \frac{x}{r}\), it follows that \(\cos \theta\) is positive wherever \(x > 0\), which is in Quadrants I and IV. (Remember, \(r\) is always positive.)

Where will \(\sin \theta\) be positive? _______________________________________________________________________

Where will \(\tan \theta\) be positive? _______________________________________________________________________  

So... what is a trick to help me remember?

"All Stupid Teachers Cheat" can help you identify which of the main trig. functions (\(\sin\), \(\cos\), and \(\tan\)) are positive for each quadrant. Go counter-clockwise and then just remember that the trig. functions reciprocal will have the same sign (so \(\csc\) is positive in all the quadrants where \(\sin\) is positive).

Example 3: State the quadrant(s) in which \(\theta\) terminates based on the given information.

\[
\begin{align*}
a. \ \cos \theta &> 0 \\
b. \ \sin \theta &< 0, \ \cos \theta < 0 \\
c. \ \sec \theta &< 0, \ \tan \theta < 0 \\
d. \ \csc \theta &> 0, \ \cot \theta < 0
\end{align*}
\]
Example 4: Given $\tan \theta = -\frac{5}{4}$ and $\cos \theta > 0$, find the exact values of the other 5 functions.

$x = \text{______} \quad y = \text{______} \quad r = \text{______} \quad \begin{align*}
\sin \theta &= \text{______} \\
\cos \theta &= \text{______} \\
\tan \theta &= \text{______} \\
\csc \theta &= \text{______} \\
\sec \theta &= \text{______} \\
\cot \theta &= \text{______}
\end{align*}$

We will be focusing on the Unit Circle more and more over the next few days. What is a Unit Circle? Well, it is a circle that is centered at the origin and that has a radius of one unit. **What effect will that have?**

What are the coordinates for $x$ and $y$? ________________

What is the cosine of $\theta$? ________________

What is the sine of $\theta$? ________________

Hence, in the **UNIT CIRCLE**…

- the cosine of an angle is always ________________
- AND the sine of an angle is always ________________.

**Quadrantal Angles**

Now, we will focus on angles in standard position that terminate on an axis, also known as **quadrantal angles**.

What happens as the terminal side of the angle approaches the $x$-axis?

What happens when the terminal side is **ON** the $x$-axis?

What happens as the terminal side of the angle approaches the $y$-axis?

What happens when the terminal side is **ON** the $y$-axis?

Armed with this knowledge, let’s figure out the trig function values for any angle whose terminal side lies on an axis, also known as a **quadrantal angle**.

**REMEMBER...**

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\cos \theta &= \frac{x}{r} \\
\tan \theta &= \frac{y}{x} \\
\csc \theta &= \frac{r}{y} \\
\sec \theta &= \frac{r}{x} \\
\cot \theta &= \frac{x}{y}
\end{align*}
\]
Example 5: Evaluate the value of each of the six trigonometric functions for each quadrantal angle \( \theta \).

<table>
<thead>
<tr>
<th>Quadrantal Angle</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( \theta = 0^\circ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
<tr>
<td>b. ( \theta = 90^\circ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
<tr>
<td>c. ( \theta = -180^\circ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
<tr>
<td>d. ( \theta = \frac{3\pi}{2} )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
</tbody>
</table>

Now get out your “unit circle” and fill in the angle values and trig values for the quadrantal angles.

Example 6: State the quadrant(s) or axis in which \( \theta \) terminates based on the given information.

<table>
<thead>
<tr>
<th>Quadrantal Angle</th>
<th>Condition</th>
<th>Quadrant(s) or Axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( \sin \theta &gt; 0 )</td>
<td></td>
<td>1, 2</td>
</tr>
<tr>
<td>b. ( \sin \theta &gt; 0, \cos \theta &lt; 0 )</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>c. ( \csc \theta &gt; 0, \tan \theta &lt; 0 )</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>d. ( \tan \theta ) is undefined</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>e. ( \sec \theta &gt; 0, \cot \theta &lt; 0 )</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>f. ( \csc \theta ) is undefined</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>g. ( \cos \theta = 0 )</td>
<td></td>
<td>2, 3</td>
</tr>
<tr>
<td>h. ( \sin \theta = -1 )</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

PART TWO: Reference Angles – Recall from a brief introduction in PART ONE.

Example 1: Sketch the angle in standard position. Then shade in the reference angle and give its measure in degrees.

<table>
<thead>
<tr>
<th>Angle</th>
<th>Sketch</th>
<th>Reference Angle</th>
<th>Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( 120^\circ )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b. ( 240^\circ )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c. ( -72^\circ )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2: Sketch the angle in standard position. Then shade in the reference angle and give its measure in radians.

| a. \( \frac{4\pi}{3} \) | b. \( -\frac{6\pi}{5} \) | c. \( \frac{\pi}{4} \) |

Let’s change the approach. I will give you the reference angle and the quadrant that the original angle terminates in... you give the measure of an angle that has the given reference angle.

Example 3: Give the measure of an angle in degrees that has the given reference angle and that terminates in the designated quadrant.

| a. 60°; terminates in Q II | b. 30°; terminates in Q III | c. 82°; terminates in Q IV |

Example 4: Give the measure of an angle in radians that has the given reference angle and that terminates in the designated quadrant.

| a. \( \frac{\pi}{6} \); terminates in Q II | b. \( \frac{\pi}{5} \); terminates in Q III | c. \( \frac{\pi}{3} \); terminates in Q I |

Recall that we briefly saw the special right triangles in our previous unit.

Recall that we briefly saw the special right triangles in our previous unit.

Let’s find all of the angles measuring between 0° and 360° that have a reference angle of 30°, 45°, or 60°.

| Angles with 30° Reference \( \angle \) | Angles with 60° Reference \( \angle \) | Angles with 45° Reference \( \angle \) |
Now, with radians. Find all of the angles between 0 and $2\pi$ that have a reference angle of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$.

<table>
<thead>
<tr>
<th>Angles with $\frac{\pi}{6}$ Reference $\angle$</th>
<th>Angles with $\frac{\pi}{4}$ Reference $\angle$</th>
<th>Angles with $\frac{\pi}{3}$ Reference $\angle$</th>
</tr>
</thead>
</table>

Is there some kind of pattern to these angles in radian measure?

Now, let's put it all together on the front side of the Unit Circle you have been provided...

Finally, let's practice a bit more with evaluating trigonometric functions with our calculator. We did this type of problem in our previous unit, but it never hurts to revisit this skill again.

Example 5: Round each of the following to four decimal places. Watch your mode!

<table>
<thead>
<tr>
<th>a. $\sin 73^\circ$</th>
<th>b. $\cos (-263^\circ)$</th>
<th>c. $\tan \frac{15\pi}{8}$</th>
<th>d. $\csc \frac{\pi}{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d. $\sec (-54^\circ)$</td>
<td>e. $\cot (-\frac{7\pi}{5})$</td>
<td>f. $\tan 301^\circ$</td>
<td>g. $\sec \frac{13\pi}{9}$</td>
</tr>
</tbody>
</table>

Remember... the calculator only does what you instruct it to do. If you tell it to find the wrong thing, it will.

PART THREE: More with the Unit Circle... and not the Quadrantal Angles! 😊

We were briefly introduced to the Unit Circle previously as we evaluated the trigonometric values for various quadrantal angles. But not every angle is a quadrantal angle, is it? Let's start by focusing just on the first quadrant of the unit circle.
So, let’s put it all together to give us the first quadrant of the unit circle. 😊

Let’s now go back to our SEPARATE Unit Circle and fill in the remaining coordinates.

It will make your life TONS easier if you memorize the following pairs, which are reciprocals of each other.

| $\frac{1}{2}$ and __________ | $\sqrt{2}$ and __________ | $\sqrt{3}$ and __________ | $\frac{\sqrt{3}}{3}$ and __________ |

Example 1: Using reference angles and the unit circle, find the values of the 6 trigonometric functions.

a. $\frac{\pi}{4}$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

b. $\frac{5\pi}{3}$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

c. $150^\circ$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

d. $\frac{7\pi}{6}$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

e. $-135^\circ$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

f. $-60^\circ$
   
   $\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots$
   
   $\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots$

Example 2: The terminal side of an angle $\theta$ in standard position passes through $\left( \frac{3}{5}, \frac{4}{5} \right)$. Determine the exact values of the 6 trigonometric functions of the angle $\theta$.

\[
\sin \theta = \ldots \quad \cos \theta = \ldots \quad \tan \theta = \ldots
\]

\[
\csc \theta = \ldots \quad \sec \theta = \ldots \quad \cot \theta = \ldots
\]
PART FOUR: Domain and Period of Sine and Cosine

What is the DOMAIN of the sine and cosine functions? ________________________________

What is the RANGE of the sine and cosine functions? ________________________________

Do all co-terminal angles have the same trig function values? ________________

So, if we wanted to find the sine or cosine of an angle measuring 390° we would ______________________

______________________________________________________________________________

If we wanted to find the sine or cosine of an angle measuring \( \frac{11\pi}{3} \), we would ______________________

______________________________________________________________________________

As a matter of fact all angles of the form ____________________ or ____________________ will have the same sine or cosine values as \( \theta \).

Hence, we say sine and cosine are PERIODIC FUNCTIONS and have a PERIOD of 360° or 2\( \pi \) radians.

Example 1: Evaluate the exact value of the given function at the given angle.

<table>
<thead>
<tr>
<th>a. ( \cos 420° )</th>
<th>b. ( \sin \left( -\frac{7\pi}{4} \right) )</th>
<th>c. ( \cos \frac{8\pi}{3} )</th>
<th>d. ( \sin \frac{21\pi}{4} )</th>
</tr>
</thead>
</table>

Even and Odd Trigonometric Functions (Negative Angle Identities):

A function is EVEN if _______________________________________________________________.

A function is ODD if ________________________________________________________________.

Which of the trig function(s) are odd? Let's look at \( \theta \) and \( -\theta \).

\[
\begin{align*}
\cos \theta &= \\
\sin \theta &= \\
\tan \theta &= 
\end{align*}
\]

Therefore, the odd trig. functions are ___________________________ and the even trig. functions are ___________________________.
### Example 2:
Use the value of the given trigonometric function to evaluate the indicated function.

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta = \frac{4}{5} )</td>
<td>( \sin(-\theta) ) and ( \csc(-\theta) )</td>
<td>( \cos(-\theta) ) and ( \sec(-\theta) ).</td>
</tr>
<tr>
<td>( \cos \theta = \frac{2}{3} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Example 3:
Use a calculator to evaluate the value to 4 decimal places.

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \frac{\pi}{5} )</td>
<td>( \cot 1.7 )</td>
<td>( \csc 2.3 )</td>
</tr>
<tr>
<td>( \sec(-2.4) )</td>
<td>( \sec \frac{4\pi}{3} )</td>
<td>( \tan 267^\circ )</td>
</tr>
</tbody>
</table>

### PART FIVE:
Given a function value, can you give the angle(s) that have that function value?

Let's look at \( \sin \theta = \frac{1}{2} \). How many angles have that sine value? ________________________________

OK, then how many between \( 0^\circ \) and \( 360^\circ \)? _____________________________

What are they? ____________________________________________

### Example 1:
Solve for \( \theta \), where \( 0^\circ \leq \theta < 360^\circ \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \theta = -\frac{1}{2} )</td>
<td>( \tan \theta = 1 )</td>
<td></td>
</tr>
</tbody>
</table>

### Example 2:
Solve for \( \theta \), where \( 0 \leq \theta < 2\pi \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \csc \theta = \frac{2\sqrt{3}}{3} )</td>
<td>( \cot \theta = -\sqrt{3} )</td>
<td></td>
</tr>
</tbody>
</table>