Section 2.5 – Zeros of Polynomial Functions

Rational Zeros (Roots) Theorem:

Let \( f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 \) (where \( a_0 \neq 0 \)) be a polynomial function in standard form that has integral coefficients. THEN if the nonzero rational number \( \frac{p}{q} \) in lowest terms is a zero of \( p(x) \), \( p \) must be a factor of the constant term \( a_0 \) AND \( q \) must be a factor of the leading coefficient \( a_n \).

Example 1: Determine the possible rational zeros of each polynomial.

a. \( f(x) = x^4 - 7x^3 - 3x^2 + 2x + 12 \)

b. \( g(x) = 6x^4 - 3x^3 + x^2 - 10x + 15 \)

What kind of polynomial equation(s) can I solve easily? ________________________

Example 2: Determine the exact values of the zeroes of each polynomial. Use your calculator to get started.

a. \( p(x) = 7x^3 + 18x^2 - 97x - 60 \)

b. \( f(x) = 22x^4 + 65x^3 - 20x^2 - 45x + 18 \)
Fundamental Theorem of Algebra:
Every polynomial function of positive degree with complex coefficients has at least one complex zero.

NOTE: The zero may be a real number since ANY real number $r$ can be expressed as the complex number $r + 0i$.

Number of Roots Theorem:
If $f(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $f(x) = 0$ has exactly $n$ roots, where roots are counted according to their multiplicity.

Conjugate Pair Theorem:
If $f(x) = 0$ is a polynomial equation real coefficients, then when $a + bi$ is a root, $a - bi$ is also a root. If $f(x) = 0$ is a polynomial equation rational coefficients, then when $m + \sqrt{n}$ is a root, $m - \sqrt{n}$ is also a root.

Example 3: Determine the zeros of each polynomial.

<table>
<thead>
<tr>
<th>a. $f(x) = x^4 - x^3 + x^2 - 3x - 6$</th>
<th>b. $g(x) = x^5 + 5x^4 - 8x^3 - 40x^2$</th>
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<tbody>
<tr>
<td>c. $f(x) = 9x^4 + 131x^3 + 183x^2 + 9x - 52$</td>
<td>d. $g(x) = 2x^4 + 11x^3 + 2x^2 - 65x - 100$</td>
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Example 4: Determine the remaining zeros of the polynomial given one of the zeros. Explain how you arrived at your answer.

\[ f(x) = x^4 - 13x^3 + 61x^2 - 127x + 78; \quad 3 + 2i \]

Example 5: Use your calculator to answer the following questions about the given polynomial function.

\[ f(x) = x^4 - 7x^3 - 46x^2 + 14x + 88 \]

Possible Rational Roots: ________________________________________________

Actual Rational Roots: _____________

# of Real Roots: _____________

# of Irrational Roots: _____________

Irrational Roots: ________________ (Use the zero feature on your graphing calculator to estimate.)

# of Imaginary Roots: _____________

Imaginary Roots: _____________

Example 6: Sketch a possible graph with the following conditions.

a. A fourth degree polynomial with a positive leading coefficient, two distinct negative real zeros greater than -5, and one positive real zero less than 4 with a multiplicity of 2.

b. A fifth degree polynomial with a negative leading coefficient, two distinct negative real zeros greater than -4 (one with multiplicity 3), and one positive real zero less than 3.