Pre-Calculus Notes

Section 9.5 - Binomial Theorem

DAY ONE: Several expansions of binomials may offer clues to a general formula for \((a+b)^n\).

\[
\begin{align*}
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{align*}
\]

Let's look for patterns that may be generalized to a formula for \((a+b)^n\).

Variables:

1. First term contains \(a^n\), 2nd term \(a^{n-1}b\), 3rd term \(a^{n-2}b^2\), ..., last term \(b^n\)
2. First term contains \(b^n\), 2nd term \(a b^{n-1}\), 3rd term \(a^2 b^{n-2}\), ..., last term \(a^n\)
3. For each term, sum of the exponents is \(n\)

How to get the coefficients? USE PASCAL'S TRIANGLE!

\[
\begin{array}{cccccccc}
1 & & & & & & & 1 \\
1 & 1 & & & & & 1 & 1 \\
1 & 2 & 1 & & & & 1 & 2 & 1 \\
1 & 3 & 3 & 1 & & & 1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 & & 1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 & 1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
\]

Example 1) Use the formula for \((a+b)^4\) to find the expansion for \((2x+y)^4\):

\[
(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\]

\[
a = 2x \quad b = y
\]

\[
(2x+y)^4 = (2x)^4 + 4(2x)^3(y) + 6(2x)^2(y)^2 + 4(2x)(y)^3 + (y)^4
\]

\[
= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4
\]
Example 2) Use the formula for \((a+b)^3\) to find the expansion for \((2x+5y)^3\).

\[
(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\]

\[
a = 2x \quad b = -5y
\]

\[
(2x+5y)^3 = 1(2x)^3 + 3(2x)^2(-5y) + 3(2x)(-5y)^2 + 1(-5y)^3
\]

\[
= 8x^3 - 60x^2y + 150xy^2 - 125y^3
\]

Example 3) Use the formula for \((a+b)^n\) to find the first four terms in the expansion of \((x^2 - y^3)^6\).

\[
(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots
\]

\[
a = x^2 \quad b = -y^3
\]

\[
(x^2 - y^3)^6 = 1(x^2)^6 + \binom{6}{1}(x^2)^5(-y^3) + \binom{6}{2}(x^2)^4(-y^3)^2 + \binom{6}{3}(x^2)^3(-y^3)^3 + \ldots
\]

\[
= x^{12} - 6x^{10}y^3 + 15x^8y^6 - 20x^6y^9 + \ldots
\]

Now, do you remember the difference between permutations \((P)\) and combinations \((C)\)?

Ex. 4) \(7P_3 = \,
\]

\[
\frac{7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}
\]

Do you remember how to do this using your calculator? math, PRB, #2

Ex. 5) How many ways can 10 contestants place 1st, 2nd, and 3rd in a contest? order matters

\[
10P_3 = 720
\]

\[
\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 720
\]

Therefore...

\[
\binom{n}{k} = \frac{n!}{(n-k)!}
\]

Ex. 6) \(7C_3 = \,
\]

\[
\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{7!}{4! \cdot 3!} = \frac{7!}{(7-3)! \cdot 3!}
\]

Do you remember how to do this using your calculator? math, PRB, #3

Ex. 7) How many ways can a committee of 3 people be chosen from 10 people? order does not matter

\[
10C_3 = 120
\]

\[
\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120
\]

Therefore...

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Examples:

a. Find $15 \binom{8}{5}$.

\[6435\]

b. Find $\binom{20}{6} = 20 \binom{6}{6} = 38760$

REMEMBER... $\binom{n}{0} = 1$ AND $\binom{n}{n} = 1$. (From Algebra II)

REMEMBER THESE?

\[
\begin{align*}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6
\end{align*}
\]

YUCK! Let's see if we can figure out a way to do the last one WITHOUT doing the expansions before it.

Put these values in your calculator in $y = \binom{n}{x}$ and check the table values starting at $x = 0$:

\[
\begin{align*}
y &= 2 \binom{8}{x} & 1 & 2 & 1 & \text{are the first 3 values.} \\
y &= 3 \binom{8}{x} & 1 & 3 & 3 & 1 & \text{are the first 4 values.} \\
y &= 4 \binom{8}{x} & 1 & 4 & 6 & 4 & 1 & \text{are the first 5 values.} \\
y &= 5 \binom{8}{x} & 1 & 5 & 10 & 10 & 5 & 1 & \text{are the first 6 values.} \\
y &= 6 \binom{8}{x} & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \text{are the first 7 values.}
\end{align*}
\]

What do you notice? **These are the coefficients for the expansion $(a+b)^n$**

Could you use this to write the first 6 terms of $(a+b)^{10}$ above?

\[
(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a b^{n-1} + \binom{n}{n}b^n
\]
Examples:

a. Find \((x+y)^9\) using the Binomial Theorem.

\[
(x+y)^9 = 1 \cdot x^9 + 9 \cdot x^8 y + 36 \cdot x^7 y^2 + 84 \cdot x^6 y^3 + 126 \cdot x^5 y^4 + \ldots
\]

Coefficients:

\[
y = \binom{9}{x} \quad \text{start w/ } x = 0
\]

b. Find the expansion for \((2x-3y)^4\).

\[
(2x-3y)^4 = 1 \cdot (2x)^4 + 4 \cdot (2x)^3 (-3y) + 6 \cdot (2x)^2 (-3y)^2 + 4 \cdot (2x) (-3y)^3 + 1 \cdot (-3y)^4
\]

\[
= 16x^4 - 96x^3 y + 216x^2 y^3 - 216xy^3 + 81y^4
\]

Could we generate the coefficients using our calculator, \(y = \), and table? Yes!

\[
y = \left( \begin{array}{c}
4 \cdot \binom{4}{x} \\
2 \cdot \binom{4}{x-1} \\
-3 \cdot \binom{4}{x-2}
\end{array} \right)
\]

Add up to 4

Add up to 8

c. Find the first 4 terms of \((x-2y)^8\).

\[
(x-2y)^8 = 1 \cdot x^8 - 16 \cdot x^7 y + 112 \cdot x^6 y^2 - 448 \cdot x^5 y^3 + \ldots
\]

Coefficients:

\[
y = \binom{8}{x} \quad \text{start w/ } x = 0
\]

Day Two:

For the following expansion, what do you notice about relationship between the number of a term and the exponents and coefficient.

\[
(a+b)^n = \binom{n}{1} a^n b^1 + \binom{n}{2} a^{n-1} b^2 + \binom{n}{3} a^{n-2} b^3 + \binom{n}{4} a^{n-3} b^4 + \binom{n}{5} a^{n-4} b^5
\]

Thus, for \((a+b)^n\)...

Last exponent in each term is always one less than the # of the term.

Exponents of the two terms always add to the power of the expansion, or \(n\).

Coefficient is always \(n \binom{x}{\text{last exponent}}\) or \(n \binom{n-x}{1\text{st exponent}}\).

because the coefficients are "Symmetric"
Examples:

<table>
<thead>
<tr>
<th>a. Find the 9th term of ((a+b)^{11}).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(11 \binom{8}{3} \cdot (a)^8 \cdot (b)^3)</td>
</tr>
<tr>
<td>One less</td>
</tr>
<tr>
<td>(= 165a^8b^3)</td>
</tr>
</tbody>
</table>

b. Find the 4th term of \((2c+3y^2)^7\).

| \(7 \binom{3}{4} \cdot (2c)^4 \cdot (3y^2)^3\) |
| Have to add to 7 |
| \(= 15120c^4y^6\) |

c. Find the 6th term of \((a+2b)^8\).

| \(8 \binom{5}{3} \cdot (a)^5 \cdot (2b)^3\) |
| One less |
| \(= 1792a^5b^3\) |

d. Complete the term for \(b^5\) in \((a-b)^{10}\).

| \(10 \binom{3}{7} \cdot (a)^3 \cdot (-b)^7\) |
| Have to add to 10 |
| \(= -120a^7b^3\) |

e. Find the coefficient of the term \(a^6b^5\) in the expansion of \((3a-2b)^{11}\).

| \(11 \binom{5}{3} \cdot (3a)^5 \cdot (-2b)^5\) |
| Same |
| \(= -1771536a^5b^5\) |

f. The probability of a baseball player getting a hit during any given time at bat is \(\frac{1}{4}\). To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term \(\binom{10}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^7\) in the expansion of \(\left(\frac{1}{4} + \frac{3}{4}\right)^{10}\).

| Plug in to calculator |
| \(\approx 0.250282\) |