**Pre-Calculus Notes**

### Introduction to Conic Sections

The conic sections result from intersecting a plane with a double cone, as shown in the figure. There are three distinct families of conic sections:

- the ellipse (including the circle)
- the parabola (with one branch)
- the hyperbola (with two branches)

Before we work with these various conic sections, we are going to practice our algebra skills in order to get the equations into standard form. We will also become familiar with identifying the conic section from its equation in standard form.

### Standard Form of Conic Sections:

<table>
<thead>
<tr>
<th>Conic</th>
<th>Equation in Standard Form</th>
<th>Generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIRCLE</td>
<td>((x-h)^2 + (y-k)^2 = r^2)</td>
<td>sum of (x^2) and (y^2) w/ same coefficients</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ex. (3x^2 + 3y^2 = 27)</td>
</tr>
<tr>
<td>ELLIPSE</td>
<td>(\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1)</td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td>(\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1)</td>
<td>sum of (x^2) and (y^2) w/ different coefficients</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ex. (5x^2 + 10y^2 = 20)</td>
</tr>
<tr>
<td>HYPERBOLA</td>
<td>(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1)</td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td>(\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1)</td>
<td>difference of (x^2) and (y^2) or (y^2) and (x^2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ex. (3x^2 - 3y^2 = 10) or (4y^2 - x^2 = 12)</td>
</tr>
<tr>
<td>PARABOLA</td>
<td>(4c(y-k) = (x-h)^2)</td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td>(4c(x-h) = (y-k)^2)</td>
<td>only (x^2) or only (y^2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NOT BOTH</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ex. (x^2 + y = 5) or (y^2 - 2y = x)</td>
</tr>
</tbody>
</table>

### Example 1: Identify each conic from its equation.

a. \((2x)^2 + 5y^2 - 6y + 16 = y^2\)
   
   \(4x^2 + 5y^2 - 6y + 16 - y^2 = 0\)
   
   \(4x^2 + 4y^2 - 6y + 16 = 0\) **Circle**

b. \(5x^2 + 3y^2 - 6x + 10 = 8x^2\)
   
   \(5x^2 + 3y^2 - 6x + 10 - 8x^2 = 0\)
   
   \(3y^2 - 3x^2 - 6x + 10 = 0\) **Hyperbola**

c. \(6x^2 - y^2 + 3x - 2y = 6x^2\)
   
   \(6x^2 - y^2 + 3x - 2y - 6x^2 = 0\)
   
   \(-y^2 + 3x - 2y = 0\) **Parabola**

d. \(5x^2 + (3y)^2 - 6x + 4y + 2 = 3y^2\)
   
   \(5x^2 + 9y^2 - 6x + 4y + 2 - 3y^2 = 0\)
   
   \(5x^2 + 9y^2 - 6x + 4y + 2 = 0\) **Ellipse**
Recall the standard forms for the conic sections:

<table>
<thead>
<tr>
<th>CONIC</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CIRCLE</td>
<td>((x-h)^2 + (y-k)^2 = r^2)</td>
</tr>
</tbody>
</table>
| ELLIPSE  | \[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{OR} \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1
\] |
| HYPERBOLA| \[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{OR} \quad \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1
\] |
| PARABOLA | \[4c(y-k) = (x-h)^2 \quad \text{OR} \quad 4c(x-h) = (y-k)^2\] |

Example 2: Identify each conic section. Then rewrite in standard form by completing the square.

a. \(x^2 - 4x + y^2 + 2y + 1 = 0\)
   \[
   x^2 - 4x + \underline{4} + y^2 + 2y + \underline{1} = -1 + \underline{4} + \underline{1}
   \]
   \[(x - 2)^2 + (y + 1)^2 = 4\]

b. \(9x^2 + 54x - 4y^2 + 40y - 55 = 0\)
   \[
   9x^2 + 54x - 4y^2 + 40y = 55
   \]
   \[9\left(x^2 + 6x + \underline{9}\right) - 4\left(y^2 - 10y + \underline{25}\right) = 55 + 9\left(\underline{9}\right) - 4\left(\underline{25}\right)
   \]
   \[9(x + 3)^2 - 4(y - 5)^2 = 36\]
   \[
   \frac{(x+3)^2}{4} - \frac{(y-5)^2}{9} = 1
   \]

b. \(9x^2 + 54x - 4y^2 + 40y - 55 = 0\)
   \[
   9x^2 + 54x - 4y^2 + 40y = 55
   \]
   \[9\left(x^2 + 6x + \underline{9}\right) - 4\left(y^2 - 10y + \underline{25}\right) = 55 + 9\left(\underline{9}\right) - 4\left(\underline{25}\right)
   \]
   \[9(x + 3)^2 - 4(y - 5)^2 = 36\]
   \[
   \frac{(x+3)^2}{4} - \frac{(y-5)^2}{9} = 1
   \]

c. \(x^2 - 6x + 8y - 7 = 0\)
   \[
   x^2 - 6x + \underline{9} = -8y + 7 + \underline{9}
   \]
   \[(x - 3)^2 = -8y + 16\]
   \[(x - 3)^2 = -8(y - 2)\]

Hyperbola: difference of \(x^2\) and \(y^2\)

Circle: \(x^2 + y^2\), same coefficients

Parabola: only \(x^2\)